Grade 9/10 Math Circles

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Probability II

Last time, we discussed how to formalize the idea of probability and covered some useful rules for calculating probabilities. We will build on those ideas today by discussing independent events and conditional probability.

Motivating Problem

Matching Socks: Suppose you have a bin with socks of different colours. If you take 2 socks from that bin randomly, how likely are you to get a matching pair?

Independent Events

Some events in a sample space may be related to each other. The idea of independence allows us to quantify which events influence each other. Intuitively, two events are independent if knowing whether one happened does change the probability that the other event happened.

For instance, hair colour is independent of which hand someone uses to write. Knowing someone has brown hair does not give any information about how likely they are to be right-handed.

Conversely, liking math and liking physics are dependent. Because the subjects are related, someone who likes math is more likely to also enjoy physics.

Definition 1. Two events A and B are *independent* if

$$P(A \cap B) = P(A) \cdot P(B)$$

If two events are not independent, we say they are *dependent*.

Example 1. Which of the following are independent?

1. Rolling a die multiple times
2. Drawing multiple cards, without replacement
3. A and B, given that $P(A) = \frac{2}{5}$, $P(B) = \frac{1}{4}$ and $P(A \cap B) = \frac{3}{20}$

Solution:

1. Independent
2. Dependent: knowing which card is removed gives information
3. Dependent: $P(A) \cdot P(B) = \frac{2}{5} \cdot \frac{1}{4} = \frac{2}{20}$. This is not equal to $P(A \cap B)$

Independence also allows us to think about the outcomes of multiple experiments at the same time. These could be entirely separate (rolling a die and drawing a card) or consecutive (flipping a coin multiple times). It is particularly useful when it is difficult to count all the possible ways for the events to occur.

Example 2. Suppose you roll a die and draw a card. Find the probability that

1. you roll a 1 and draw a spade
2. you roll an even number and draw an ace

Solution: Rolling a die and drawing a card are independent.

1. $P(1 \cap \text{spade}) = P(1) \cdot P(\text{spade}) = \frac{1}{6} \cdot \frac{1}{4} = \frac{1}{24}$
2. $P(\text{even} \cap \text{ace}) = P(\text{even}) \cdot P(\text{ace}) = \frac{1}{2} \cdot \frac{1}{13} = \frac{1}{26}$

Exercise 1. Suppose you roll a six-sided die twice. Find the probability that you roll:

1. an even number followed by an odd number
2. a 1 followed by any other number

Exercise 2. Suppose there are 2 red balls, 3 green balls, and 5 blue balls in a bin. You randomly choose two balls, replacing the first before drawing another. Find the probability that:

1. the first ball is red and the second is green
2. both balls are blue

Conditional Probability

When working with dependent events, it is useful to know how the probability of an event changes once we know information about other events. This includes problems where we select items without replacement.

Definition 2. The *conditional probability* of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}, \text{ given that } P(B) \neq 0.$$

Example 3. Suppose you roll two six-sided dice. What is the probability that their sum is at least 10? What is the probability that their sum is at least 10 given that the first roll is a 5?

Solution: Let's say $F =$ the first roll is a 5 and $S =$ the rolls sum to at least 10.

First, let's find the probability of S. There are 36 possible outcomes of rolling two dice. There are six rolls that give sums of at least 10 (4/6, 5/5, 5/6, 6/4, 6/5 and 6/6). So, $P(S) = 6/36 = 1/6$.

If the first roll is a 5, then the second roll must be a 5 or 6 for the sum to be at least 10. So, $P(F \cap S)$ will be the probability of rolling a 5 and then rolling a 5 or 6. Since rolls are independent,

$$P(F \cap S) = P(\{5\}) \cdot P(\{5, 6\}) = 1/6 \cdot 2/6 = 2/36$$

So,

$$P(S|F) = \frac{P(F \cap S)}{P(F)} = \frac{2/36}{1/6} = 1/3$$

As we might expect, rolling a 5 first increases the probability that the roll will sum to at least 10.

Let's check that this definition makes sense by calculating the same values in a more intuitive way:

If we already know that the first roll is a 5, we just need to calculate the probability that the second roll is a 5 or 6. We know this probability is indeed 1/3.

Example 4. A gumball machine contains 3 colours of gumballs. If you purchase a gumball, there is a 20% chance it will be blue, a 40% chance it will be pink, and a 40% chance it will be red. Someone really likes red gumballs, so they broke into the machine and stole all the red ones. What is the chances of getting a blue gumball now?

Solution: Let B = the event that you get a blue gumball and R = the event that you get a red gumball. We need to find $P\left(B|R^C\right)$.

To do so, we will need that $P\left(R^C\right) = 1 - P(R) = 1 - 0.4 = 0.6$.

Also, notice that you can only get one colour of gumball at a time, so $P\left(B \cap R^C\right) = P(B)$.

Using the conditional probability formula, we have

$$P\left(B|R^C\right) = \frac{P\left(B \cap R^C\right)}{P\left(R^C\right)} = \frac{P(B)}{P\left(R^C\right)} = \frac{0.2}{0.6} = \frac{1}{3}$$

As we might expect, the probability of getting a blue gumball increases.

Exercise 3. Suppose you roll two six-sided dice. What is the probability that their sum is at least 10 given that you rolled doubles?

Exercise 4. Suppose B is a subset of A. What is $P(A|B)$?

Fun Fact: A and B are independent exactly when $P(A|B) = P(A)$.

Notice that if they are independent, then the following equality holds:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(A) \cdot P(B)}{P(B)} = P(A)$$

This makes sense since independent events should not affect each others probabilities.

We can also use conditional probability to help us solve the sock problem. In particular, it will be helpful to use the following rearrangement of the definition of conditional probability:

$$P(A \cap B) = P(B) \cdot P(A|B)$$

Example 5. Suppose you have a bin with 6 black socks, 10 red socks, and 4 white socks. If you select two socks from the bin, how likely are you to get a pair of red socks?

Solution: Let's say $R1$ = first sock is red and $R2$ = second sock is red. We want to find $P(R1 \cap R2)$.

It would be very difficult to count how many total possibilities there are, as well as how many ways you can get two red socks. Instead, we can use the above formula and calculate

$$P(R1 \cap R2) = P(R1) \cdot P(R2|R1)$$

For the first sock, there are 10 red out of 20 total, so $P(R1) = 10/20$.

After taking out a red sock, there are 9 red out of 19 total, so $P(R2|R1) = 9/19$.

$$P(R1 \cap R2) = \frac{10}{20} \cdot \frac{9}{19} = \frac{90}{380} \approx 24\%$$

Exercise 5. Continuing from the previous example, find the probability of getting any matching pair of socks.

In the sock problem, we split up the probability of getting a pair based on the colour of that pair. We can also split up the probability of an event based on whether another event occurs - the probability of an event E is the probability that both E and F occur plus the probability that E occurs but F does not. (We can say this since exactly one of F and F^C will occur.) That is,

$$P(E) = P(E \cap F) + P\left(E \cap F^C\right)$$

We can use the trick from the sock problem to rewrite this statement as

$$P(E) = P(E|F) \cdot P(F) + P\left(E|F^C\right) \cdot P\left(F^C\right)$$

This rule is particularly useful when an event depends on another event, so it is easier to calculate conditional probabilities.

Example 6. Suppose that it rains 30% of all days. There is traffic 50% of the days that it is raining but only 10% of the days it is not raining. What is the probability that there is traffic on a random day?

Solution: Let R be the event that it rains and let T be the event that there is traffic. We know $P(T|R) = 5/10$, $P\left(T|R^C\right) = 1/10$, and $P(R) = 3/10$. The complement rule says $P\left(R^C\right) = 7/10$. So,

$$P(T) = P(T|R) \cdot P(R) + P\left(T|R^C\right) \cdot P\left(R^C\right) = \frac{5}{10} \cdot \frac{3}{10} + \frac{1}{10} \cdot \frac{7}{10} = \frac{22}{100}$$

Exercise 6. Many people who like board games also like video games. Suppose 70% of people play board games, and of those people 50% also play video games. Only 20% of people who don't play board games play video games. How likely is a random person to play video games?

We will use this rule alongside the following theorem.

Bayes' Theorem

This theorem tells us about the relationship between $P(A|B)$ and $P(B|A)$.

Bayes' Theorem: Let A and B be two events. Then,

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \text{ given that } P(B) \neq 0$$

Example 7. Suppose that 1/4 of people have a cat, 1/3 of people have a fish, and 1/6 of people with a cat also have a fish. If you find out that someone has a fish, what is the probability that they also have a cat?

Solution: Let C mean the person has a cat and F mean the person has a fish. We want to find $P(C|F)$.

We know that $P(C) = 1/4$, $P(F) = 1/3$, and $P(F|C) = 1/6$. Using Bayes' Theorem,

$$P(C|F) = \frac{P(F|C)P(C)}{P(F)} = \frac{\frac{1}{6}\cdot\frac{1}{4}}{\frac{1}{3}} = \frac{3}{24}$$

Example 8. Suppose there are two coins: a regular coin and a coin biased to land on heads 70% of the time. You randomly select a coin and toss it once.

1. What is the probability that the coin lands on heads?
2. If it lands on heads, what is the probability that it is biased?

Solution: Let B mean the coin is biased and H mean the coin lands on heads. Note that there is a 50% chance of choosing the biased coin, so $P(B) = P\left(B^C\right) = 0.5$.

1. The probability that the coin lands on heads depends on whether or not it is biased. So, we will split the probability of landing on heads based on whether or not it is biased.

$$P(H) = P(H|B)\cdot P(B) + P\left(H|B^C\right)\cdot P\left(B^C\right) = \frac{7}{10}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{12}{20}$$

2. Using Bayes' Theorem,

$$P(B|H) = \frac{P(H|B)\cdot P(B)}{P(H)} = \frac{\frac{7}{10}\cdot\frac{1}{2}}{\frac{12}{20}} = \frac{7}{12} \approx 0.58$$

Exercise 7. Suppose that 80% of people like cookies, 40% of people like brownies, and 90% of people who like brownies also like cookies. If your friend tells you that they like cookies, how likely is it that they also like brownies?